BOOK OF ABSTRACTS

SINGULARITIES IN THE NORTH: 2025 MINI-WORKSHOP

Mid Sweden University, Sundsvall

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Speakers

Alexandre Fernandes	Federal University of Ceará (Brazil)
Andreas Lind	Mid Sweden University (Sweden)
Anna Denkowska	Krakow University of Economics (Poland)
Aurélio Menegon	Mid Sweden University (Sweden)
Boris Shapiro	Stockholm University (Sweden)
Edson Sampaio	Federal University of Ceará (Brazil)
Juan José Nuño Ballesteros	University of Valencia (Spain)
Maciej Denkowski	Jagiellonian University (Poland)
Thais Maria Dalbelo	Federal University of São Carlos (Brazil)
Zbigniew Jelonek	Polish Academy of Science (Poland)

1. **Alexandre Fernandes**

Title: Lipschitz geometry of singularities

Abstract: In 1985, T. Mostowski proved a striking result: any complex algebraic family of subsets of Cn has only finitely many representatives up to bi-Lipschitz homeomorphism, with respect to the outer metric. Later, in 1997, A. Parusinski extended this finiteness phenomenon to the real setting, showing that any subanalytic family of subsets of R^n also admits only finitely many bi-Lipschitz types. However, these finiteness theorems do not explain how to classify the sets up to bi-Lipschitz homeomorphism. Roughly speaking, they guarantee the existence of discrete invariants that could, in principle, distinguish between the types, but they do not provide explicit classification tools. In this talk, I will introduce the Lipschitz geometry of singularities and present some key results in the field, focusing on the classification of real and complex singularities in low dimensions.

2. **Andreas Lind**

Title: Density property -- the definition and beyond

Abstract: Given a complex manifold M , we say that M has the density property if the Lie algebra of globally integrable holomorphic vector fields is dense (in the compact-open topology) in the set of all holomorphic vector fields. This property was introduced by Dror Varolin in the beginning of 2000s. The property gets even more interesting when looking at affine complex manifolds, i.e. manifolds given by polynomial equations. In the latter, we say that M has the algebraic density property if the Lie algebra of all globally integrable algebraic vector fields coincides with the set of all algebraic vector fields. A remarkable fact is that the algebraic density property implies the density property. In this talk we will give some examples of manifolds with the (algebraic) density property, and give some properties of these manifolds. Furthermore, we will see how the density property can be introduced on singular spaces and give some glimps into ongoing research together with A. Menegon.

3. **Anna Denkowska**

Title: The Kuratowski convergence of medial axes and conflict sets

Abstract: This is a joint work with Adam Białożyt and Maciej Denkowski from the Jagiellonian University. The medial axis of a closed, nonempty, proper subset X of R^n is the set of points x of R^n for which the Euclidean distance to X is realised in more than one point. It is a central concept in pattern recognition. It has been known for a long time that the medial axis is highly unstable under deformations: the medial axis of a planar circle is its central point, but even the smallest `protuberance' on the circle leads to the medial axis becoming a whole segment. However, this approach consists in looking at the initial and the final steps only - with nothing in between, so to say. We adopt another, natural, point of view: we see the deformation as a continuous process that we do not even require to be smooth. This lets us have some insight into what is happening to the medial axis and eventually leads to a rather surprising stability result. We extend it further to the case of conflict sets. Interestingly, this notion has direct applications in the financial markets analysis and is especially useful when combined with evolution in time. In particular, the dynamics of sudden changes in the insurance sector can be detected via the dynamics of the changes in data which is reflected in the evolution of the corresponding conflict sets or Voronoi diagrams (as this is the discrete case). The semi-continuity results we obtained provide thus also new tools for the study of systemic or climate risk.

4. **Aurélio Menegon**

Title: Perplex Functions and Real Singularities

Abstract: While complex analytic techniques offer powerful tools in the study of singularities, they often fall short when dealing with objects defined by non-holomorphic functions. In this talk, we present a class of algebraic structures on $\mathbb{R}^2$ — the perplex numbers — that extend the complex numbers in a simple yet flexible way. These fields allow us to define a generalized notion of differentiability, as well as analogues of the Cauchy-Riemann equations. We show how this framework makes it possible to construct Milnor fibrations and study the topology of real singularities.

5. **Boris Shapiro**

Title: Return of the plane evolute (joint with R.Piene and C.Riener)

Abstract: We consider the evolutes of plane real-algebraic curves and discuss some of their complex and real-algebraic properties. In particular, for a given degree $d \geqslant 2$, we provide lower bounds for the following four numerical invariants:
(1) the maximal number of times a real line can intersect the evolute of a real-algebraic curve of degree d;
(2) the maximal number of real cusps which can occur on the evolute of a real-algebraic curve of degree d;
(3) the maximal number of crunodes which can occur on the dual curve to the evolute of a real-algebraic curve of degree d;
(4) the maximal number of crunodes which can occur on the evolute of a real-algebraic curve of degree d.

6. **Edson Sampaio**

Title: TBA
Abstract: TBA

7. **Juan José Nuño Ballesteros**

Title: A general Lê-Greuel formula

Abstract: The Lê-Greuel formula is a classical result that give us an iterative method to compute the Milnor number of an isolated complete intersection singularity (ICIS). We will discuss about some extensions of this formula to the general case where we consider a pair of functions f,g on a complex analytic variety X. The idea is to give a formula for the difference between the Euler characteristics of the general fibre of f and the general fibre of the pair (f,g) (provided it admits a fibration). This is a joint work in progress with Lê and Seade.

8. **Maciej Denkowski**

Title: Reaching of singularities

Abstract: One of the most interesting features of the medial axis M of a set X is how it ` approaches' the set X, i.e. what points of X lie in the closure of M. By an old result of Nash, M stays away from points at which X defines a C^2-smooth germ. However, not all the C^2-singularities are reached by M. In the talk I will present several results concerning the characterisation of the singularities reached by the medial axis obtained in collaboration with Adam Białożyt, Lev Birbrair and Dominik Bysiewicz, and involving a generalisation of the curvature radius, a special notion of superquadraticity, and some Lipschitz properties.

9. **Thais Maria Dalbelo**

Title: Whitney equisingularity for families of hypersurfaces in toric varieties

Abstract: In one of their works, C. Eyral and M. Oka provide conditions for a family F_t of functions in $\mathbb{C}^n$ with not necessarily isolated singularities to ensure that the associated family of hypersurfaces $F_t^{-1}(0)$ is Whitney equisingular. In this paper, we establish conditions for a family ft of functions so that the associated family of hypersurfaces $f_t^{-1}(0)$ is Whitney equisingular, now within the context of toric varieties with an arbitrary singular set. This is a joint work with Danilo da Nóbrega Santos.

10. Zbigniew Jelonek

Title: Metric version of the Zariski Multiplicity Conjecture is true for multiplicity two

Abstract: We show that if two algebraic (n − 1)-dimensional cones P, $R \subset \mathbb{C}^n$ with isolated singularities are homeomorphic, then they have the same degree. We also prove that if two algebraic (n − 1)-dimensional cones P, $R \subset \mathbb{C}^n$ are ambient homeomorphic, then their bases B_p and B_r have the same Euler characteristic. As an application, we show that two bi-Lipschitz equivalent Brieskorn–Pham hypersurfaces have the same multiplicities at 0. As the second application, we show our main result: if $(X, 0)$, $(Y, 0) \subset (\mathbb{C}^n, 0)$ are germs of analytic hypersurfaces which are ambient bi-Lipschitz equivalent and $m_0(X) = 2$, then also $m_0(Y) = 2$. At the end of the paper, we also give some application of our results to the Arnold–Vassiliev Problem. Joing work with Alexandre Fernandes and José Edson Sampaio.