Mittuniversitetet
MID SWEDEN UNIVERSITY

Försättsblad Prov Original

Kurskod	Provkod	Tentamensdatum
DT049A	T101	2018-10-31

Kursnamn	Datateknik AV, Sannolikhetslära och stokastiska processer
Provnamn	Tentamen
Ort	Sundsvall
Termin	H18
Ämne	Datateknik

Exam of Probability and Random Processes
DT049A

8:00 - 13:00, October 31, 2018

Contact: Patrik Österberg (010-142 86 14)

INSTRUCTIONS
This is a open book and closed notes exam.
Aids allowed:

1. Robert G. Gallager *Stochastic Processes: Theory for Applications*

All answers must be thoroughly explained.
Be careful and mark all your solution papers with both your identification number and which question you are answering. Write clearly.
Maximum points: 36
Good Luck!

1. **Random Variables**
   (a) (3 points) Show that if the random variables X and Y are uncorrelated, then the variance of their sum equals the sum of their variances.
   (b) (3 points) Let X and Y be independent random variables having the following probability distributions:
   $$p_X(0) = p_X(1) = 1/2$$
   $$p_Y(-1) = P_Y(1) = 1/2$$
   Let $Z = XY$ and show that X and Z are uncorrelated but dependent.

   Total for Question 1: 6

2. **Poisson Processes.** Consider a generic Poisson process with exponentially-distributed interarrival times X_i, so that
$$f_{X_i}(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

with $i = 1, 2, 3..., N$. The PDF $f_{S_N}(t)$ of the random variable arising from the convolution of N-exponential random variables is called Erlang distribution. Then:

(a) (3 points) Derive the expression of the Erlang PDF $f_{S_N}(t)$ with direct calculation of the convolution. (*Hint*: start from $f_{S_1}(t)$ and use induction. Remember that the convolution of two functions $f(t)$ and $g(t)$ is defined as $f(t) * g(t) \triangleq \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau$)

(b) (1 point) Discuss the physical meaning of $f_{S_N}(t)$ for different values of N.

(c) (2 points) Calculate the moment generating function (MGF) for the exponential distribution.

(d) (2 points) Use the properties of the MGF to derive the MGF of the Erlang distribution and explain how to derive mean and variance of the Erlang distribution from the MGF.

Total for Question 2: 8

3. **Finite-State Markov Chains.**

(a) (1 point) Find the transition probability matrix P for the directed graph in Figure 1. For which value of the parameter a is the directed graph in Figure 1 a Markov chain?

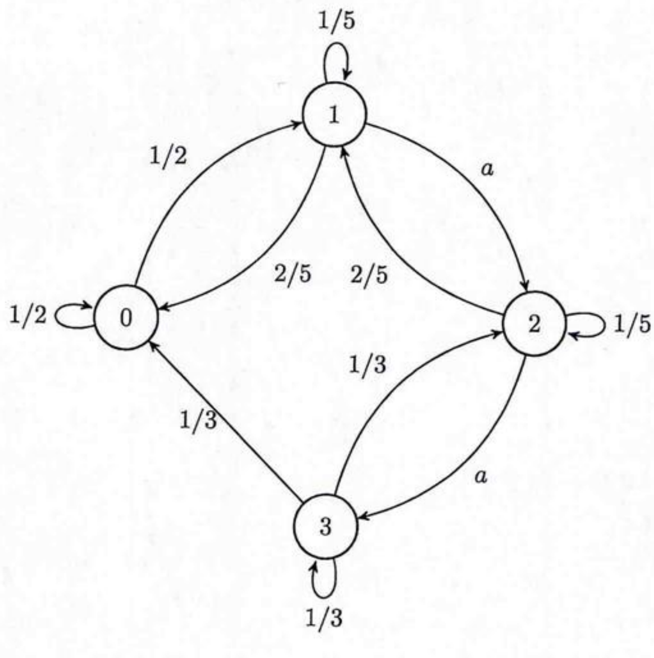

Figure 1: Finite-state Markov Chain

(b) (2 points) Classify the states in the Markov chain in *recurrent* and *transient* states. Is state 3 a periodic or aperiodic state?

(c) (2 points) Given that the chain starts with $X_0 = 3$, find the probability that $X_2 = 1$.(i.e. the probability that we move from state 3 to state 1 in two steps)

(d) (3 points) Given that the chain starts with $X_0 = 0$, find the expected first-passage time to the state 3.

Total for Question 3: 8

4. **Markov Processes.** A small bookie shop has room for at most two costumers. Potential customers arrive at a Poisson rate of 3 costumers per hour; they enter if there is room and are turned away, to never return, otherwise. The bookie serves the admitted costumers in order (FCFS), requiring an exponentially distributed time of mean 15 minutes per costumer. (*Hint.* recognize that the number of customer in the bookie shop form a Birth-death process)

   (a) (3 points) What is the average number of customers in the bookie shop?

   (b) (2 points) What is the proportion of potential customers that are turned away from the shop?

   (c) (3 points) What is the expected queuing time for a customer? What is the expected waiting time for a customer?

Total for Question 4: 8

5. **Decision Theory.** Consider the general communication scheme in Figure 2. A digital source is generating the bit-stream $\bar{S} = \{s_0, s_1, ..., s_i\}$ as a sequence of bits (e.g., 01011...). Each bit s_i is modulated using a binary amplitude shift keying (ASK) modulation, meaning that the transmitted signal x_i for each $i \in \mathbb{N}$ is

$$x_i = \begin{cases} +A & \text{if } s_i = 1 \\ -A & \text{if } s_i = 0 \end{cases}$$

with A positive real number. The signal is then transmitted over an additive white Gaussian noise (AWGN) channel and received as $y_i = x_i + n_i$, with n_i realization of the Gaussian noise affecting the i-th bit. Assume that the noise has known probability distribution $n_i \sim \mathcal{N}(\mu, \sigma^2)$.

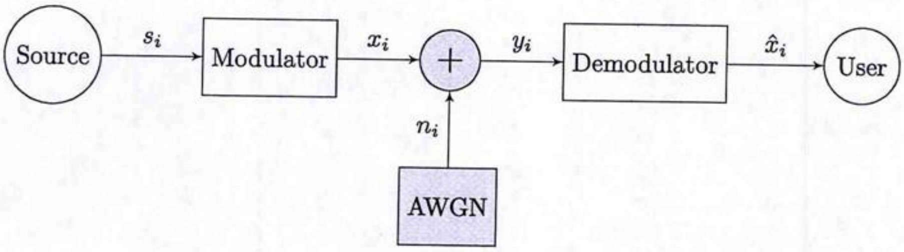

Figure 2: Communication scheme (AWGN channel). In gray the blocks representing the communication channel.

   (a) (3 points) Find the expression of the log-likelihood ratio $\Lambda_L(y)$. Assume that there is no a-priori knowledge on the probability of incoming bits.

   (b) (2 points) Find the optimal decision threshold η_M and the maximum-a-posteriori (MAP) decision criterion to be used in the demodulator.

   (c) (1 point) Is $\Lambda_L(y)$ a *sufficient statistic*? Motivate your answer.

Total for Question 5: 6